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PAPERS

BUCKLING STRESSES FOR FLAT PLATES AND SECTIONS

By Elbridge Z. Stowell,[1] George J. Heimerl,[2] Charles Libove,[3] Jun. ASCE, and Eugene E. Lundquist,[4] Assoc. M. ASCE

Synopsis

The results of extensive studies in the buckling of flat unstiffened plates conducted in the field of aeronautics during and since World War II are surveyed in this paper. The presentation of a broad perspective, rather than a detailed picture, is intended. Therefore only a few important loadings are considered. For single plates, these loadings are compression, shear, and their combination, and for the integral flat-plate sections, compression only.

The buckling stresses for various types of plate under these several loadings are given in the form of a nondimensional chart for the elastic stress range. The use of the stress-strain curve to calculate buckling stresses in the plastic range is explained. A correlation between buckling strength and maximum strength is shown for the integral flat-plate sections. Good agreement is shown between theory and experiment in a few key cases. The theoretical and experimental techniques underlying the buckling investigation are discussed briefly in the appendixes.

The good agreement shown between theory and experiment indicates that the present status of the problem of calculating flat-plate buckling strengths over the entire range of stress is now satisfactory from an engineering viewpoint.

Note.—Written comments are invited for publication; the last discussion should be submitted by January 1, 1951.

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Introduction

The use of lightweight structures of thin-sheet metal for aircraft has stimulated much research on the buckling of plates. Considerable basic progress has been made in this field of engineering during and since World War II. The use of thin-walled members in civil engineering structures makes such progress of interest to civil engineers as well as to aeronautical engineers. The purpose of this paper, therefore, is to bring part of the recent developments (mostly that part due to the Structures Research Division of the National Advisory Committee for Aeronautics (NACA)) more directly to the attention of the former group.

The importance of knowing buckling stress lies in the fact that buckling represents the beginning of a radically different regime in the behavior of a structure. In an originally flat plate, buckling is the theoretically sudden appearance of small deflections perpendicular to the plane of the plate. In practice, however, the slight initial deviations from flatness result in deflections from the beginning of loading which accelerate in growth as the theoretical buckling load for the flat plate is approached. The post-buckling regime is characterized by reduced stiffness and a wrinkling whose physical (and sometimes psychological) effects may be undesirable. The physical processes that eventually lead to the failure of a plate are initiated at buckling, and for some structures, a knowledge of the buckling stress is a key to the calculation of the additional strength beyond buckling. Since buckling is not always equivalent to failure, appreciable economies in design may often be effected by permitting the plate to buckle and taking advantage of its post-buckling strength. In aircraft design, for example, standard practice frequently permits buckling within the range of working stresses. The philosophy behind such design is similar to the philosophy of limit design proposed for civil engineering structures.[5,6,7]

The problem chosen for discussion is that of calculating the buckling stresses of flat unstiffened plates and integral flat-plate sections (integral assemblies of such plates). The objective of this paper is to convey a broad perspective of the problem, rather than an exhaustive treatment. For this reason, only a few important loading conditions are considered. For single plates, these loadings are compression, shear, and combined compression and shear; for sections, only compression. The buckling stresses for these cases are given in the form of theoretical charts. Both plastic and elastic buckling are considered. Sections on correlation with experiment and on maximum strength beyond the buckling load are also included. The theories underlying the calculation of the charts are indicated briefly in Appendix I, in which reference is also made to the literature on plate-buckling theory. The experimental techniques are described in Appendix II.

Notation.—The letter symbols in this paper are defined where they first appear.

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[5] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, p. 638.
[6] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948.
[7] Review by F. R. Shanley of "Theory of Limit Design," by J. A. Van den Broek, *Aeronautical Engineering Review*, Vol. 7, No. 4, April, 1948, p. 73.

Elastic Buckling

Form of Presentation of Theoretical Results.—A theoretical solution for the elastic buckling stress of a plate under compression load can be expressed by a numerical coefficient k_c (in which subscript c denotes "compression"), from which the buckling stress σ can be calculated by use of the formula[8]

$$\sigma = k_c \frac{\pi^2 E}{12 (1 - \mu^2)} \left(\frac{t}{b}\right)^2 \text{.......... (1)}$$

in which: E is Young's modulus; μ is Poisson's ratio; t is the plate thickness; and b is the plate width. Similarly, the shear buckling stress τ is obtained

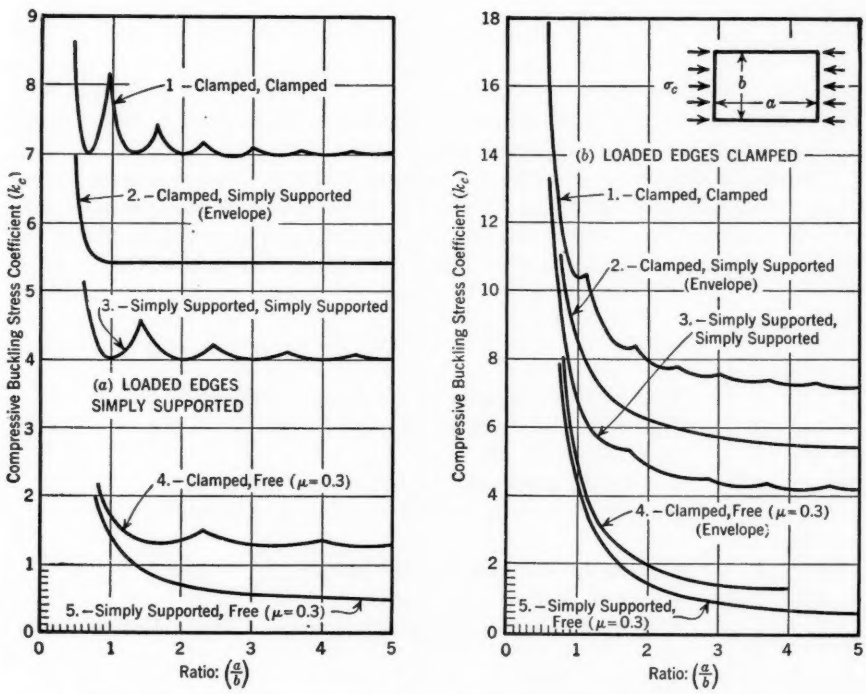


Fig. 1.—Value of Compressive Buckling Stress Coefficient for Flat Plates

by substituting a numerical coefficient k_s (subscript s denoting "shear buckling stress") in the formula

$$\tau = k_s \frac{\pi^2 E}{12 (1 - \mu^2)} \left(\frac{t}{b}\right)^2 \text{.......... (2)}$$

For given boundary conditions, the coefficients k_c and k_s generally depend upon the length-to-width ratio of the plate and can be plotted as a function of this ratio.

[8] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 331.

When both types of stress are present, the combinations that will cause buckling can be defined by combinations of values of the stress coefficients k_c and k_s to be used in Eqs. 1 and 2. For given boundary conditions and length-to-width ratio, a so-called "interaction curve" of k_c against k_s can be plotted. The interaction curves can also be plotted in terms of stress ratios R_c and R_s,[8] in which R_c is the ratio of the compressive stress σ_{cm} present at buckling under combined loading to the compressive stress σ_c required for buckling under simple compression loading; and R_s is defined similarly for shear. In equation form, the stress ratios can be defined as

$$R_c = \frac{\sigma_{cm}}{\sigma} \quad (3)$$

and

$$R_s = \frac{\tau_{cm}}{\tau} \quad (4)$$

The reduction of interaction curves to stress-ratio form often reveals a basic similarity in shape among different curves.

Results for Compression.—The inset diagram in Fig. 1 shows a plate loaded along breadth b on both edges. Values of k_c for plates in compression with various combinations of simple support, clamping, and freedom along the unloaded edges are summarized in Fig. 1, the respective authorities being as follows:

Curve	Authority
Fig. 1(*a*):	
1	Eugene E. Lundquist and Elbridge Z. Stowell[9]
2	Harry N. Hill,[10] Assoc. M. ASCE
3	G. H. Bryan[11]
4	Eugene E. Lundquist and Elbridge Z. Stowell[9]
5	Eugene E. Lundquist and Elbridge Z. Stowell[12]
Fig. 1(*b*):	
1	Charles Libove and Manuel Stein[13]
2	Harry N. Hill[10]
3	S. Timoshenko[8] and Charles Libove and Manuel Stein[13]
4	Harry N. Hill[10]
5	Harry N. Hill[10]

The caption "clamped, simply supported" on Curve 2, Fig. 1(*a*) (as an example) means that one unloaded edge is clamped and the other is simply supported. Coefficient k_c is plotted for each combination of edge conditions as a function of the length-to-width ratio a/b of the plate.

[9] "Critical Compressive Stress for Flat Rectangular Plates Supported Along all Edges and Elastically Restrained Against Rotation Along the Unloaded Edges," by Eugene E. Lundquist and Elbridge Z. Stowell, *Report No. 733*, NACA, 1942.

[10] "Chart for Critical Compressive Stress of Flat Rectangular Plates," by H. N. Hill, *Technical Note No. 773*, NACA, 1940.

[11] *Proceedings*, London Mathematical Society, Vol. 22, 1891, p. 54.

[12] "Critical Compressive Stress for Outstanding Flanges," by Eugene E. Lundquist and Elbridge Z. Stowell, *Report No. 734*, NACA, 1942.

[13] "Charts for Critical Combinations of Longitudinal and Transverse Direct Stress for Flat Rectangular Plates," by Charles Libove and Manuel Stein, *Wartime Report No. L-224*, NACA, 1946, pp. 321–401.

The boundary conditions of simple support and clamping are seldom exactly realized in practice. Usually there is some degree of elastic restraint against rotation along a supported edge, such as those symbolized in Fig. 2. Values of the compressive buckling stress coefficient k_c for each of the conditions

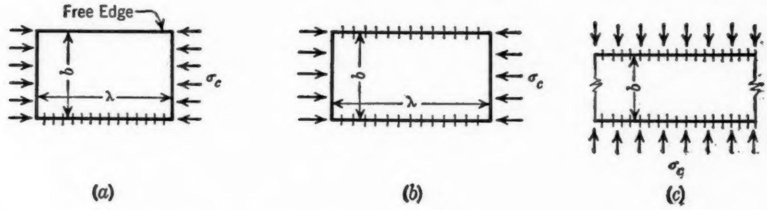


Fig. 2

represented in Fig. 2 are given by the curves in Figs. 3 and 4, respectively. For a longitudinally compressed outstanding flange[12] (Fig. 3); for a longitudinally compressed long plate with both long edges equally restrained against

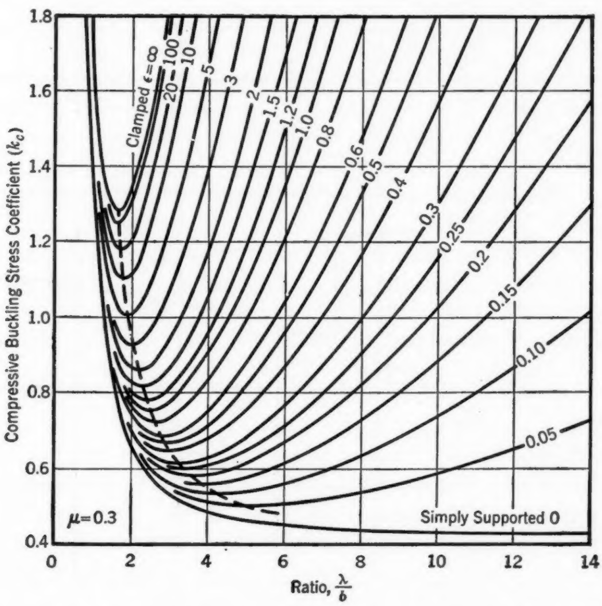
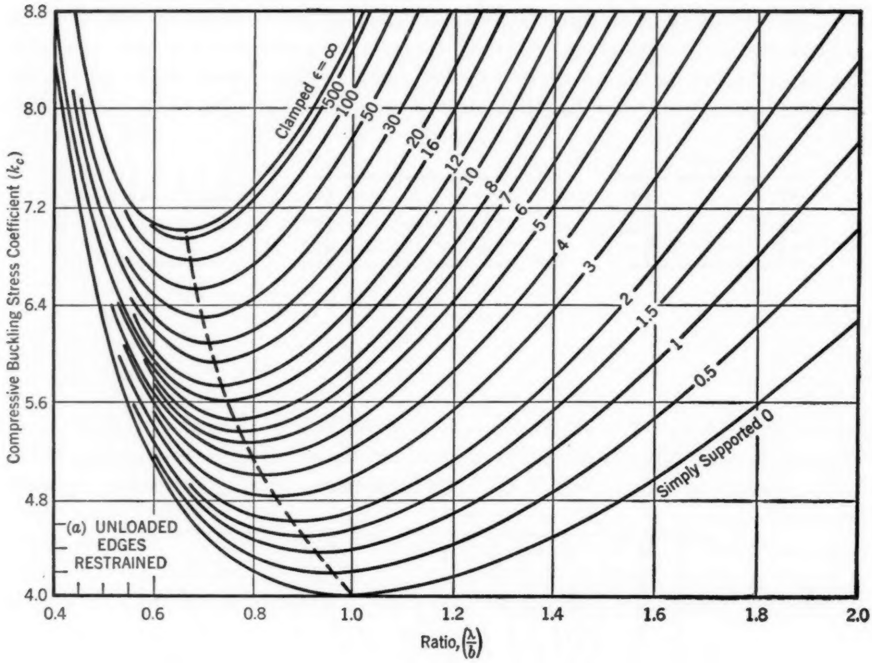


Fig. 3.—Compressive Buckling Stress Coefficient k_c for a Flange With its Supported Edge Restrained Elastically Against Rotation ($\mu = 0.3$)

rotation[10] (Fig. 4(a)); and for the same plate transversely compressed[14] (Fig. 4(b)).

In Figs. 3 and 4 the magnitude of the elastic rotational restraint appears in the parameter ϵ, which is essentially the ratio of restraint stiffness to plate

[14] "A Method for Determining the Column Curve from Tests of Columns with Equal Restraints Against Rotation on the Ends," by Eugene E. Lundquist, Carl A. Rossman, and John C. Houbolt, *Technical Note No. 903*, NACA, 1943.



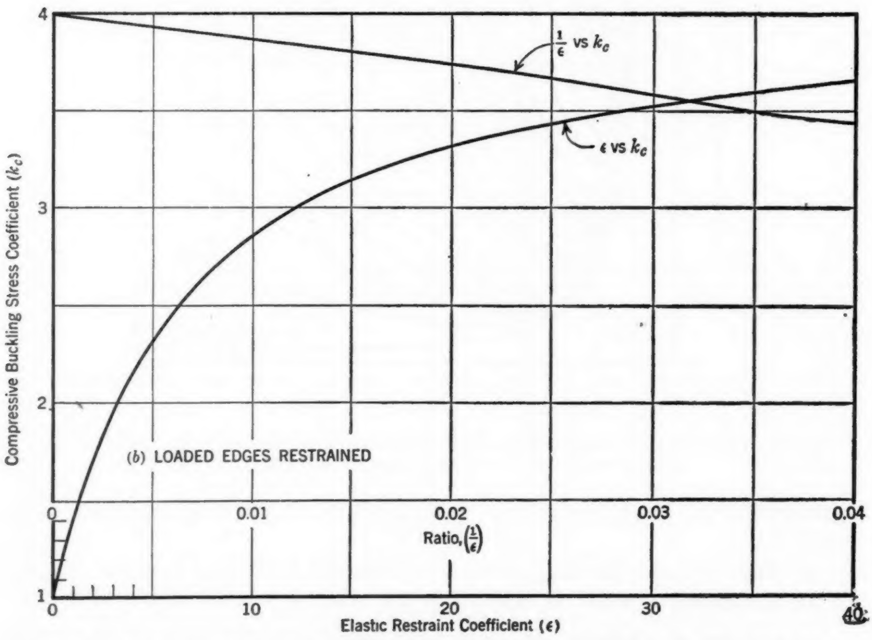


FIG. 4.—COMPRESSIVE BUCKLING STRESS COEFFICIENT FOR A LONG PLATE WITH A PAIR OF MUTUALLY OPPOSITE EDGES HAVING EQUAL ELASTIC RESTRAINT AGAINST ROTATION

stiffness and is defined[9] by the equation

$$\epsilon = \frac{4\,S_o}{S_f/b} \quad \text{(5)}$$

in which: S_o is the stiffness of the elastic restraining medium, expressed as moment per unit length per quarter radian of rotation, assumed constant along the edge of the plate; and the plate flexural stiffness is

$$S_f = \frac{E\,t^3}{12\,(1 - \mu^2)} \quad \text{(6)}$$

For simply supported edges, $S_o = \epsilon = 0$, and for clamped edges, $S_o = \epsilon = \infty$.

In Figs. 3 and 4(*a*), k_c is given not only as a function of ϵ but also as a function of λ, the buckle half-wave length or distance between successive nodes. (For the case of Fig. 4(*b*), λ is infinity.) The half-wave length λ was included as a parameter in Figs. 3 and 4(*a*) because the value of ϵ will depend upon λ for many types of elastic restraint, and also upon k_c if the restraining medium carries a load proportional to that on the plate. (The evaluation of ϵ in terms of λ and k_c for several types of elastic restraining medium is discussed subsequently.) Since λ and k_c are not known in advance, neither is ϵ. For a given plate and a given restraining medium, therefore, trial-and-error computation must be made. Mutually consistent sets of values of ϵ and λ must be assumed until a minimum value of k_c is obtained. A finite plate with simply supported edges will buckle with its length a divided into an integral number of half waves. The governing value of λ, therefore, must be sought in the sequence $a/1$, $a/2$, $a/3$, . . . and is that term which gives the lowest values of k_c.

For the academic case of a restraining medium in which rotation of one section does not influence rotations at adjacent sections (such as a series of discrete coil springs) ϵ will be independent of λ. Then, for an infinitely long plate, the buckle pattern will adjust itself to the value of λ at the minimum point of the appropriate ϵ-curve, and the plate will buckle at the corresponding stress.

Results for Shear.—Values of k_s for plates in shear with all edges either simply supported[15] or clamped[16] are given in Fig. 5(*a*). In addition, the value of k_s is given for a square plate with one pair of opposite edges clamped and the other pair simply supported.[17] For these latter boundary conditions the buckling stress will approach either the buckling stress of a plate with all edges clamped or all edges simply supported as the plate becomes more and more rectangular, depending upon which pair of edges becomes the longer.

The values of k_s for a long plate with both side edges having equal elastic restraint against rotation are given in Fig. 5(*b*),[18] which is similar to Fig. 4(*a*)

[15] "Buckling Stresses of Simply Supported Rectangular Plates in Shear," by Manuel Stein and John Neff, *Technical Note No. 1222*, NACA, 1947.

[16] "Buckling Stresses of Clamped Rectangular Flat Plates in Shear," by Bernard Budiansky and Robert W. Connor, *Technical Note No. 1559*, NACA, 1948.

[17] "Die Knickung der rechteckigen Platte durch Schubkrafte," by S. Iguchi, *Ingenieur-Archiv*, Bd. IX, Heft 1, February, 1938, pp. 1–12.

[18] "Critical Shear Stress of an Infinitely Long Flat Plate with Equal Elastic Restraints Against Rotation Along the Parallel Edges," by Elbridge Z. Stowell, *Wartime Report No. L.-476*, NACA, 1943.

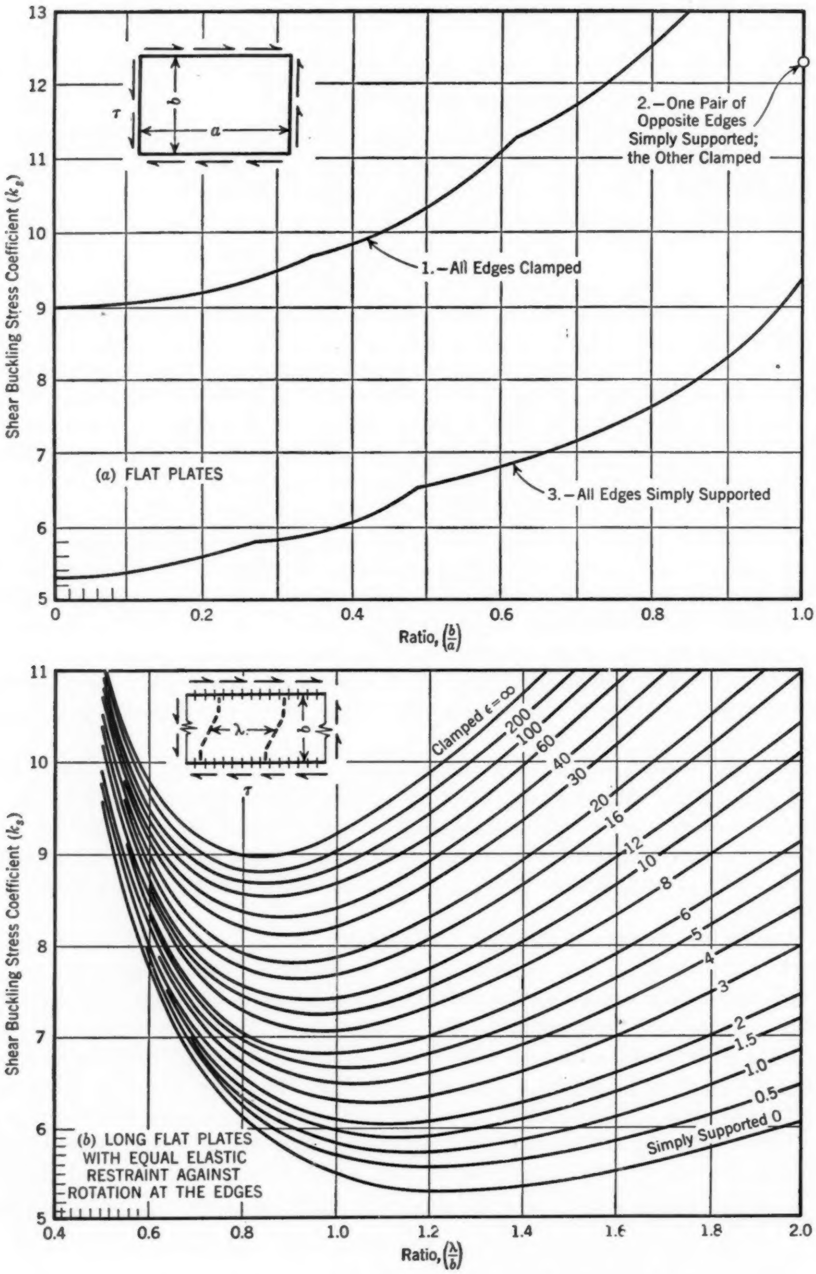


FIG. 5.—VALUE OF SHEAR BUCKLING STRESS COEFFICIENT k_s

for compression. The discussion of ϵ and λ in connection with compression also applies in Fig. 5 for shear.

Results for Combined Compression and Shear.—Combinations of values of R_c and R_s for which a simply supported plate in combined compression and shear will buckle are given in Fig. 6.[19] The results are in the form of interaction curves of R_s versus R_c for different length-to-width ratios a/b. Fig. 6 shows that when the compressive stress acts in a longitudinal direction ($a/b < 1$) the interaction curve is described by the parabola

$$R^2_s + R_c = 1 \qquad (7)$$

However, as the length-to-width ratio changes so that the compressive stress assumes a transverse nature ($a/b > 1$), the shape of the interaction curve

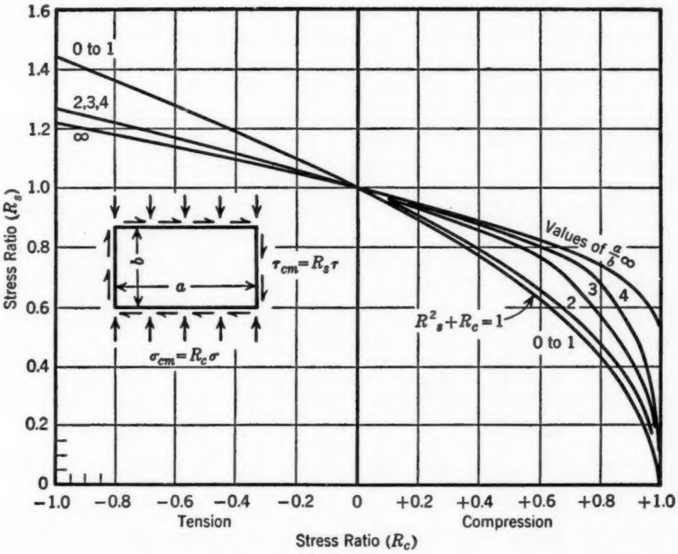


FIG. 6.—CRITICAL COMBINATIONS OF SHEAR AND DIRECT STRESS FOR A SIMPLY SUPPORTED FLAT PLATE

changes markedly. An interesting characteristic of the curve for $a/b = \infty$ is the vertical part at $R_c = 1$, which reveals that an appreciable amount of shear stress may be applied to the plate before the transverse compressive stress required for buckling is reduced.

The effect of elastic restraint against rotation is shown in Fig. 7 for a long plate with both edges equally restrained against rotation. For the case of longitudinal compression and shear in Fig. 7(a),[20] the interaction curve for all degrees of restraint between simple support and clamping can still be described by Eq. 7. Although Fig. 7(a) and Eq. 7 were developed for the type of elastic

[19] "Critical Combinations of Shear and Direct Stress for Simply Supported Rectangular Flat Plates," by S. B. Batdorf and Manuel Stein, *Technical Note No. 1223*, NACA, 1947.

[20] "Critical Stress for an Infinitely Long Flat Plate with Elastically Restrained Edges Under Combined Shear and Direct Stress," by Elbridge Z. Stowell and Edward B. Schwartz, *Wartime Report No. L-340*, NACA, 1943.

restraint whose stiffness is unaffected by buckle wave length and load on the plate, the same parabolic relationship was also found to apply when the restraint is furnished by a stiffener whose stiffness S_o is affected by both these quantities.[20] For the case of transverse compression and shear shown in Fig. 7(*b*) different curves are obtained for different values of ϵ (independent of load and buckle wave length). All the curves have the vertical sections found to exist for the simply supported plate in Fig. 7(*b*).[21]

Evaluating the Stiffness of the Elastic Restraint.—The charts for the buckling of elastically restrained plates, described in the previous sections, apply to the case in which S_o, the stiffness of the elastic restraint, is a constant along the edge. In other words, the ratio of the bending moment intensity exerted by the buckled plate on the restraining medium to the rotation produced is assumed to be the same at all points along the edge of the plate. Fortunately, this condition is satisfied either exactly or approximately in many practical cases, particularly those involving long uniform plates with uniform elastic restraint. In these cases, both the bending-moment intensity and the rotation

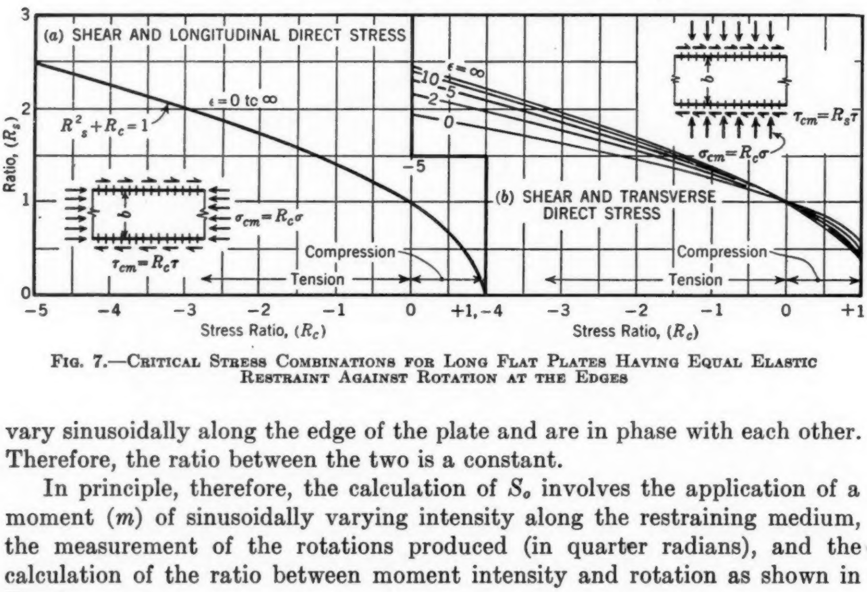


FIG. 7.—CRITICAL STRESS COMBINATIONS FOR LONG FLAT PLATES HAVING EQUAL ELASTIC RESTRAINT AGAINST ROTATION AT THE EDGES

vary sinusoidally along the edge of the plate and are in phase with each other. Therefore, the ratio between the two is a constant.

In principle, therefore, the calculation of S_o involves the application of a moment (m) of sinusoidally varying intensity along the restraining medium, the measurement of the rotations produced (in quarter radians), and the calculation of the ratio between moment intensity and rotation as shown in Fig. 8.

If the restraint is furnished by a "sturdy stiffener,"—that is, a stiffener that suffers no cross-sectional distortion when it twists—its stiffness is given by the formula[22]

$$S_o = \frac{\pi^2}{4\lambda^2}\left(GJ - \bar{\sigma} I_p + \frac{\pi^2}{\lambda^2} E C_{BT}\right) \quad \text{.............. (8)}$$

[21] "Critical Combinations of Shear and Transverse Direct Stress for an Infinitely Long Flat Plate with Edges Elastically Restrained Against Rotation," by S. B. Batdorf and J. C. Houbolt, *Report No. 847*, NACA, 1946.

[22] "Restraint Provided a Flat Rectangular Plate by a Sturdy Stiffener Along an Edge of the Plate," by Eugene E. Lundquist and Elbridge Z. Stowell, *Report No. 735*, NACA, 1941.

in which: λ is the half-wave length of a sinusoidally varying distributed twisting moment; G is the shear modulus of elasticity; J is the torsion constant of a stiffener; $G\,J$ is the torsional stiffness of a stiffener; $\bar{\sigma}$ is the average compressive stress in a stiffener; I_p is the polar moment of inertia of a stiffener sectional area about its axis of rotation; and C_{BT} is the torsion-bending constant of a stiffener sectional area about its axis of rotation at or near the edge of a plate.[23,24,25] The first term in parentheses in Eq. 8 gives the stiffness according to St. Venant; the second term gives the reduction in stiffness due to the compressive load in the stiffener; and the third term gives the stiffening effect due to the fact that the cross sections under a variable rate of twist are partly restrained against longitudinal warping.

If the plate being restrained is part of an assembly of plates, then the elastic restraining medium may itself be an adjoining plate under compression with its own far edge clamped, simply supported, free, or subjected to a moment

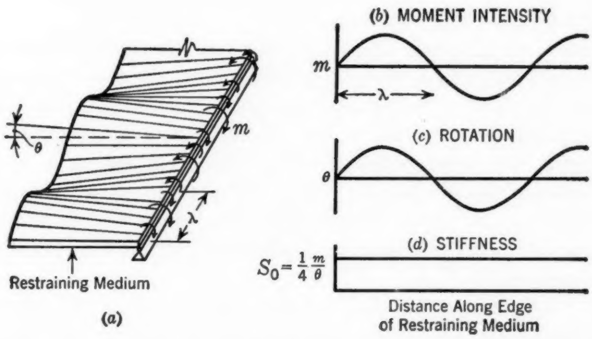


Fig. 8.—Evaluation of the Rotational Stiffness S_o of an Elastic Restraining Medium

equal and opposite to that at its near edge. In such a case, the stiffness of the restraining medium can be evaluated from tables of stiffness and carry-over factor for flat rectangular plates under compression, prepared by Wilhelmina D. Kroll in 1943.[26] In this table, the values S_o of a plate with its far edge clamped, simply supported, free, or subjected to a moment equal and opposite to that at the near edge are denoted by S, S^{II}, S^{III}, and S^{IV}, respectively, and are given as functions of the stress in the plate and the half-wave length of edge rotation.

In some cases, the far edge of the restraining plate may itself be restrained elastically against rotation by one or more plates with a combined stiffness S_1.

[23] "Torsion and Buckling of Open Sections," by Herbert Wagner, *Technical Memorandum No. 807*, NACA, 1936.

[24] "Torsion and Buckling of Open Sections," by H. Wagner and W. Pretschner, *Technical Memorandum No. 784*, NACA, 1936.

[25] "Twisting Failure of Centrally Loaded Open-Section Columns in the Elastic Range," by Robert Kappus, *Technical Memorandum No. 851*, NACA, 1938.

[26] "Tables of Stiffness and Carry-over Factor for Flat Rectangular Plates Under Compression," by W. D. Kroll, *Wartime Report No. L-398*, NACA, 1943.

In that case, the stiffness S_o of the first restraining plate may be expressed in terms of S_1 by the formula[27]

$$S_o = \frac{S^{II}}{1 - C^2\left(\frac{S_1}{S^{II} + S_1}\right)} \quad \text{(9)}$$

in which C is the carry-over factor of the first restraining plate and is tabulated with the stiffnesses by Miss Kroll;[26] and S^{II} is the stiffness that the first restraining plate would have if its far edge were simply supported. Eq. 9 is in the nature of a recurrence formula giving the stiffness S_o of any plate in terms of the restraint S_1 at its far edge. Successive application of Eq. 9 (starting with a plate for which the conditions at the far edge are known and working up to the plate under consideration) will yield the stiffness S_o to be used in conjunction with Figs. 3 and 4(a). This procedure is illustrated elsewhere.[27] In some cases, it may be necessary to assume simple support or clamping at some juncture one or two junctures away from the edge of the plate under consideration in order to shorten the process.

The k charts themselves (Figs. 3 and 4(a) for compression and Fig. 5(b) for shear) can sometimes give the stiffnesses of restraining plates. For example, Fig. 5(b) gives the external restraining stiffness S_o for which buckling will occur in shear at a given "load" k_s and wave length ratio λ/b. When buckling occurs, the total stiffness of the joint consisting of the juncture of plate and restraining medium is zero. The stiffness of the plate itself at buckling, therefore, is $-S_o$. Thus, Fig. 5(b), with the signs of the ϵ-values reversed, gives the stiffness of a long plate in shear corresponding to S^{IV} for a plate in compression.

Results for I*-Columns,* Z*-Columns, Channel-Columns, and Rectangular-Tube-Section Columns.*—By considering one of the component plates of the column to be elastically restrained by its neighbors and evaluating S_o with the aid of the stiffness tables,[26] it was possible to use Figs. 3 and 4(a) to calculate the local buckling stresses for columns of several simple cross-sectional shapes. The results are given in terms of a numerical parameter k_w in Fig. 9.[28] With k_w known, the buckling stress σ can be obtained from the formula

$$\sigma = k_w \frac{\pi^2 E}{12(1 - \mu^2)}\left(\frac{t_w}{b_w}\right)^2 \quad \text{(10)}$$

in which the subscript w refers to the web plate.

Method of Obtaining Results for Other Plate Sections.—For more general types of plate section, the joint-stiffness criterion[27] may be used to calculate the local buckling strength. This criterion states that when a plate assembly is carrying its local buckling load, the sum of the stiffnesses of all the members entering a joint is zero. To apply this principle, it is necessary to assume a series of values of buckling stress and for each one to calculate the stiffnesses

[27] "Principles of Moment Distribution Applied to Stability of Structures Composed of Bars or Plates," by Eugene E. Lundquist, Elbridge Z. Stowell, and Evan H. Schuette, *Report No. 809*, NACA, 1945, Eq. 13.

[28] "Charts for Calculation of the Critical Stress for Local Instability of Columns with I-, Z-, Channel, and Rectangular Tube Section," by W. D. Kroll, Gordon P. Fisher, and George J. Heimerl, *Wartime Report No. L-429*, NACA, 1943.

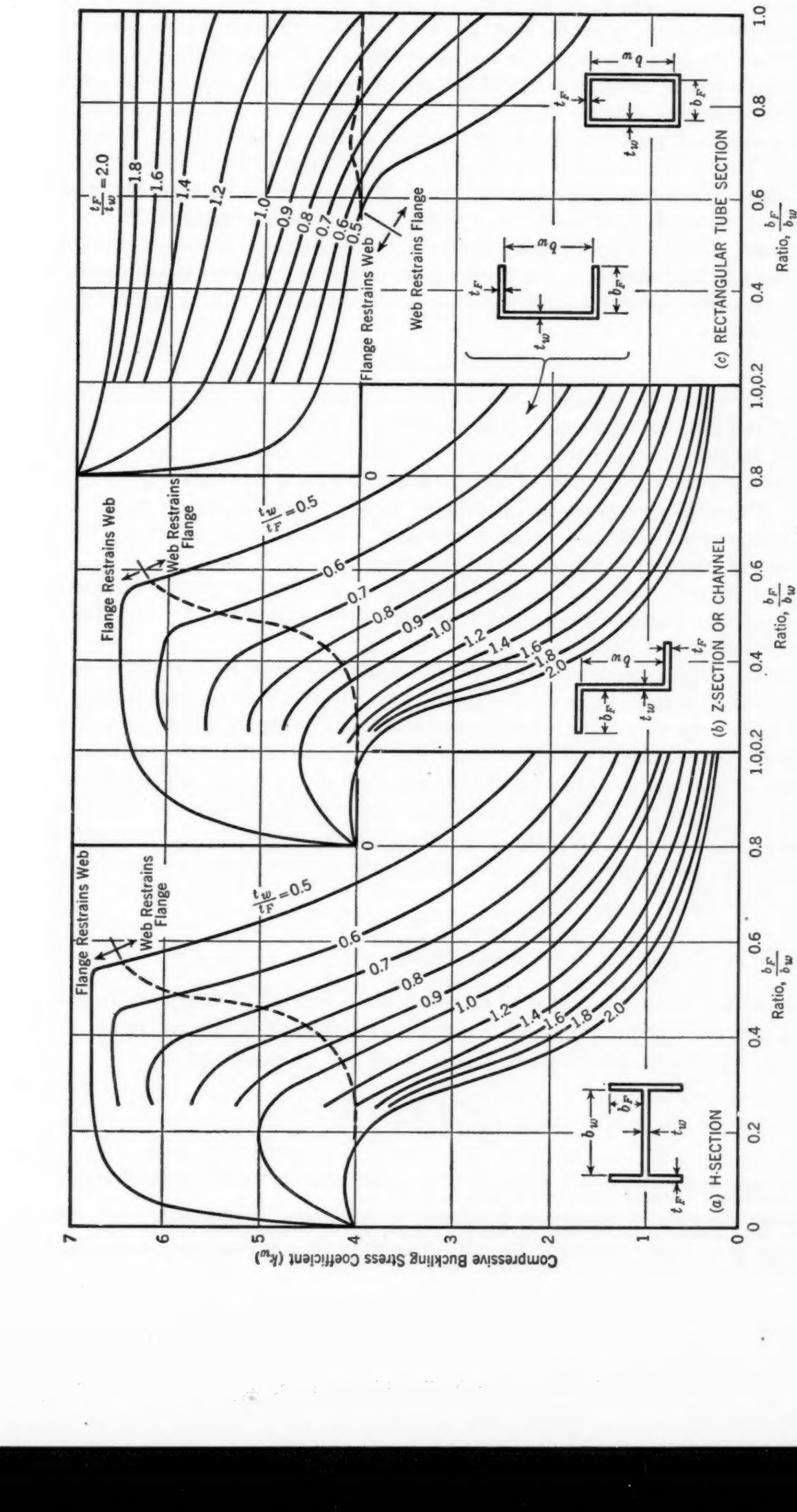


FIG. 9.—COMPRESSIVE PLATE BUCKLING STRESS COEFFICIENT k_w

of the members meeting at some particular joint by the methods described at the end of the section on single plates. The lowest stress for which the sum of the stiffnesses equals zero is the buckling stress. This procedure must be repeated for each of several assumed values of λ until a minimum buckling stress is obtained.

This procedure applies only if the corners of the section remain straight lines when local buckling occurs. In open sections having small enough lips, such as lipped-Z-section columns, the lip may translate in its own plane at buckling. Two studies of this type of buckling have been completed.[29,30]

The general condition of buckling of plate assemblies, with some account of the fact that the intersections of the plates may not remain straight, was given in an earlier publication,[31] together with numerical results for various types of sections.

Plastic Buckling

When the buckling stress is low enough to remain on the straight line section of the stress-strain curves, the buckling is called elastic and the equations and figures discussed thus far apply. When the buckling stress is high enough to lie on the curved part of the stress-strain curve, the buckling is called plastic and new factors must be introduced to allow for the departure of the stress-strain curve from the original straight elastic section at these high stresses in the plastic range.

Form of Presentation of Theoretical Results.—Data on the plastic buckling stresses for plates under single loading have been presented in two forms. Both forms are used in this paper.

The first, or classical, form stems from the concept that Eqs. 1 and 2 may be used in the plastic range if the modulus of elasticity E is replaced by a reduced modulus $\eta\, E$, η being a function of the buckling stress, the shape of the stress-strain curve, the type of loading, the length-to-width ratio and edge conditions of the plate. (In the elastic range, $\eta = 1$ and above the elastic range, $\eta < 1$.) In this form of presenting the results, a curve of η versus buckling stress is given for a plate of specified material and specified boundary conditions. In terms of η the compressive buckling stress σ_c can be calculated from the formula

$$\sigma = k_c \frac{\pi^2\, \eta_c\, E}{12\,(1 - \mu^2)} \left(\frac{t}{b}\right)^2 \qquad (11)$$

in which the subscript c is added to η to indicate a loading of compression and μ and E are the elastic values of Poisson's ratio and Young's modulus, respectively. Similarly, the shear buckling stress can be expressed as

$$\tau = k_s \frac{\pi^2\, \eta_s\, E}{12\,(1 - \mu^2)} \left(\frac{t}{b}\right)^2 \qquad (12)$$

[29] "The Local Buckling Strength of Lipped Z-Columns with Small Lip Width," by Pai C. Hu and James C. McCulloch, *Technical Note No. 1335*, NACA, 1947.

[30] "Primary Instability of Open-Section Stringers Attached to Sheet," by Samuel Levy and Wilhelmina D. Kroll, *Journal of the Aeronautical Sciences*, Vol. 15, No. 10, October, 1948, p. 581.

[31] "Theory of the Plastic Stability of Thin Plates," by P. P. Bijlaard, *Publications*, International Association for Bridge and Structural Engineering, Vol. 6, 1940–41, p. 45.

Because η_c and η_s in Eqs. 11 and 12 are themselves given as functions of the unknown buckling stresses σ and τ, a trial-and-error calculation is generally necessary to obtain the buckling stress of a given plate. This difficulty is overcome in the second or alternate form of presenting the results.

The alternate presentation is essentially a plot of actual buckling stress σ (or τ) versus elastic buckling strain $\frac{\sigma}{E\,\eta_c}\left(\text{or } \frac{\tau}{G\,\eta_s}\right)$. Such a plot is directly obtainable from the plot of σ versus η_c (or τ versus η_s) and a knowledge of E (or G). The parameters of the buckling curve just described are dimensionally those of a stress-strain curve. The stress-strain curve of the plate material may therefore be superimposed on the same set of coordinates often revealing a useful correlation.

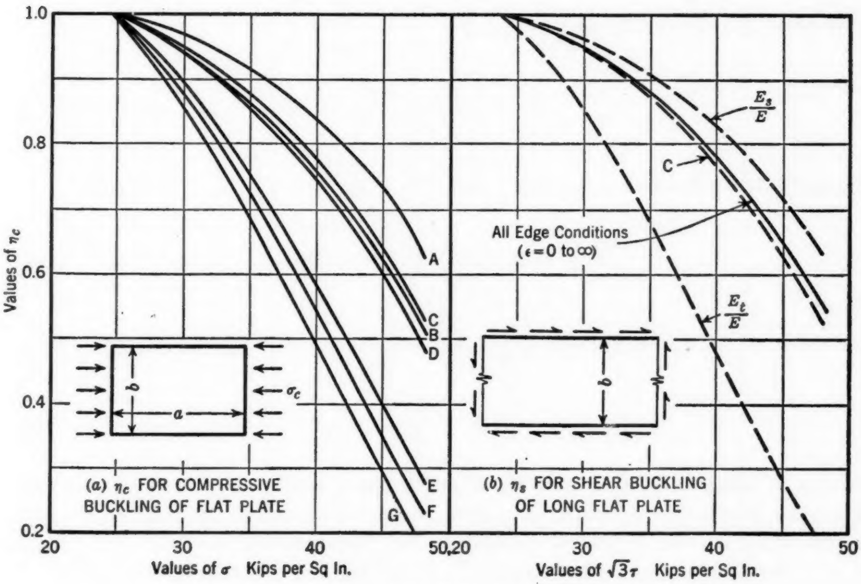


FIG. 10.—PLASTICITY REDUCTION FACTORS

For combined compression and shear, the stress ratios R_c and R_s used in the elastic range can also be used to present the results of buckling calculations in the plastic range.

Results for Compression.—Curves of η_c versus buckling stress σ are given in Fig. 10(a) for plates of 24S-T aluminum alloy. This alloy has a compressive yield stress of 46 kips per sq in. (0.2% offset). The loaded edges of the plate were simply supported, and the results for various types of plates are given in the following list[32] (see Fig. 10(a)):

Long hinged flange—

$$\eta_c = \frac{E_s}{E} \qquad \text{(13a)}$$

[32] "A Unified Theory of Plastic Buckling of Columns and Plates," by Elbridge Z. Stowell, *Report No. 898*, NACA, 1948.

long clamped flange—

$$\eta_c = \frac{E}{E}\left[0.330 + 0.670\sqrt{\frac{1}{4} + \frac{3}{4}\frac{E_t}{E_s}}\right] \qquad (13b)$$

long plate, hinged edges, simply supported—

$$\eta_c = \frac{E_s}{E}\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{4} + \frac{3}{4}\frac{E_t}{E_s}}\right] \qquad (13c)$$

long plate, clamped edges—

$$\eta_c = \frac{E_s}{E}\left[0.352 + 0.648\sqrt{\frac{1}{4} + \frac{3}{4}\frac{E_t}{E_s}}\right] \qquad (13d)$$

wide plate as a short column $\left(\frac{a}{b} \leqq 1\right)$—

$$\eta_c = \frac{1}{4}\frac{E_s}{E} + \frac{3}{4}\frac{E_t}{E} \qquad (13e)$$

square plate loaded as a column $\left(\frac{a}{b} = 1\right)$—

$$\eta_c = 0.114\frac{E_s}{E} + 0.886\frac{E_t}{E} \qquad (13f)$$

narrow plate as a long column $\left(\frac{a}{b} \geqq 1\right)$—

$$\eta_c = \frac{E_t}{E} \qquad (13g)$$

Curves of secant modulus ratio E_s/E and tangent modulus ratio E_t/E, derived from the stress-strain curve, are also given. The coincidence of these two curves with curves A and G indicates that the reduced modulus for a long hinged flange is the secant modulus E_s, and that for a narrow plate-column is the tangent modulus E_t. Eqs. 13 are the basic formulas from which the η-curves can be computed for any material.

Three of the η-curves shown in Fig. 10(a) are replotted in Fig. 11 in the more directly usable form of buckling stress σ_c versus e (elastic), the strain at which buckling would occur in an elastic material.

Results for Shear.—A curve of η versus $\sqrt{3}\,\tau$ is given for long plates of 24S-T aluminum alloy in Fig. 10(b). This curve[33] applies to all degrees of equal elastic edge restraint against rotation between simple support and clamping. For comparison, curves of E_s/E and E_t/E (obtained from the compressive stress-strain curve with σ replaced by $\sqrt{3}\,\tau$) are also given. Also shown dotted, for comparison, is curve C, of Fig. 10(a), for a compressed simply supported plate. This curve practically coincides with the curve for η_s.

[33] "Critical Shear Stress of an Infinitely Long Plate in the Plastic Region," by Elbridge Z. Stowell, *Technical Note No. 1681*, NACA, 1948.

Combined Longitudinal Compression and Shear.—Interaction curves for three long plates of 24S-T aluminum alloy are given in Fig. 12. The three plates were so chosen[34] that one would buckle in the elastic range for all stress combinations, one well in the plastic range, and one in the intermediate range. The three curves can be described approximately by the single parabola—

$$\left[R_s \frac{E_{s\tau}}{E_{sm}}\right]^2 + R_c \frac{E_{sc}}{E_{sm}} = 1 \qquad (14)$$

—in which R_s and R_c are the stress ratios previously defined in Eqs. 3 and 4, and all values of E_s are secant moduli obtained from the compressive stress-

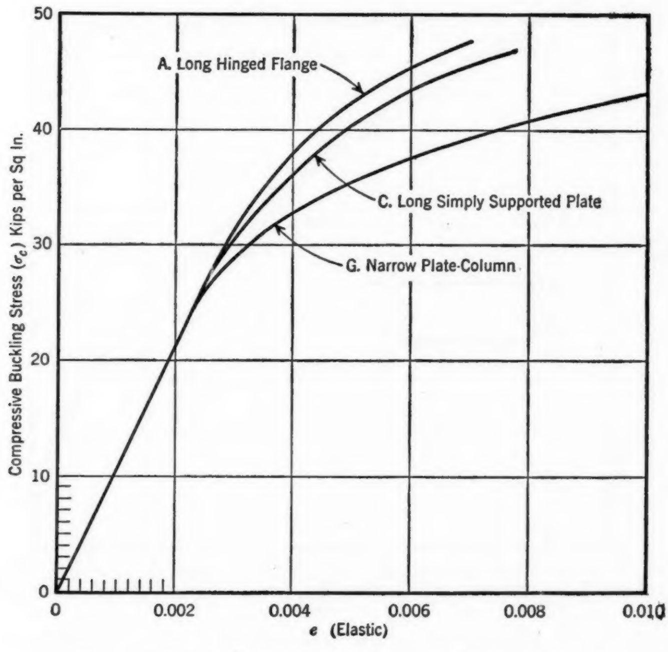


FIG. 11.—VALUES OF COMPRESSIVE PLASTIC BUCKLING STRESS

strain curve and defined as follows: E_{sm} is the secant modulus corresponding to a stress—

$$\sqrt{(\sigma_{cm})^2 + 3(\tau_{cm})^2} \qquad (15a)$$

—in which σ_{cm} and τ_{cm} are the stresses present at buckling; $E_{s\tau}$ is the secant modulus corresponding to a stress—

$$\sigma_i = \sqrt{3}\,\tau \qquad (15b)$$

[34] "Plastic Buckling of a Long Flat Plate Under Combined Shear and Longitudinal Compression," by Elbridge Z. Stowell, *Technical Note No. 1990*, NACA, 1949.

—in which τ_e is the stress required for buckling is shear alone; and E_{sc} is the secant modulus corresponding to a stress

$$\sigma_i = \sigma \qquad (15c)$$

—in which σ is the stress required for buckling in compression alone.

The parameters used in Fig. 12 were chosen to show the parabolic relationship as indicated by Eq. 14. These interaction curves can also be replotted in terms of R_c and R_s alone. Such curves would have the same end points but would otherwise lie above the curves shown. Thus the use of the elastic interaction Eq. 7 would be conservative in the plastic range.

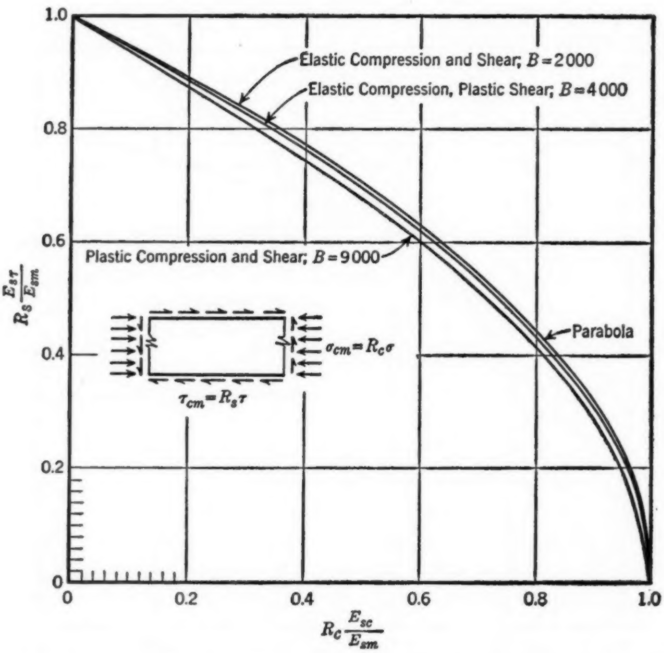


FIG. 12.—PLASTIC BUCKLING STRESSES IN COMBINED SHEAR AND COMPRESSION FOR LONG FLAT PLATES WITH SIMPLY SUPPORTED EDGES; $B=\frac{\pi^2 E\, b^2}{12(1-\mu^2)b^2}$

Results for **H**-*Section Columns.*—The basic theory for plastic buckling for single plates[32] has been applied to the more complex problem of the **H**-section column.[35] The calculated results for extruded 75S-T aluminum alloy are given in Fig. 13 as curves of buckling stress σ versus elastic buckling strain ϵ_e (elastic) for two families of cross sections $\left(\frac{t_w}{t_F} = 1; \frac{b_F}{b_w} = 0.6 \text{ to } 0.8\right)$ The compressive stress-strain curve for this material is also included in order to show how closely it follows the buckling curves.

[35] "Plastic Buckling of Extruded Composite Sections in Compression," by Elbridge Z. Stowell and Richard A. Pride, *Technical Note No. 1971*, NACA, 1949.

Comparison of Theory and Experiment

Compression.—The correlation of the compression test results with theory for both elastic and plastic buckling of the hinged flange (cruciform), the simple supported plate (square tube) and the H-section is illustrated in Fig. 14, with the test data taken from previous experiments.[32,36,37] The results are plotted in the form of experimental buckling stress σ_c versus calculated strain ϵ_c (elastic) at which buckling would occur if the material remained perfectly elastic. The compressive stress-strain curves are included for comparison. Appendix II illustrates the techniques used to detect buckling and to obtain the stress-strain curves.

In each case the agreement between theory and test results is good. For the hinged flange as represented by the cruciform section, both the theory and test results fall on the stress-strain curve, indicating that the effective modulus

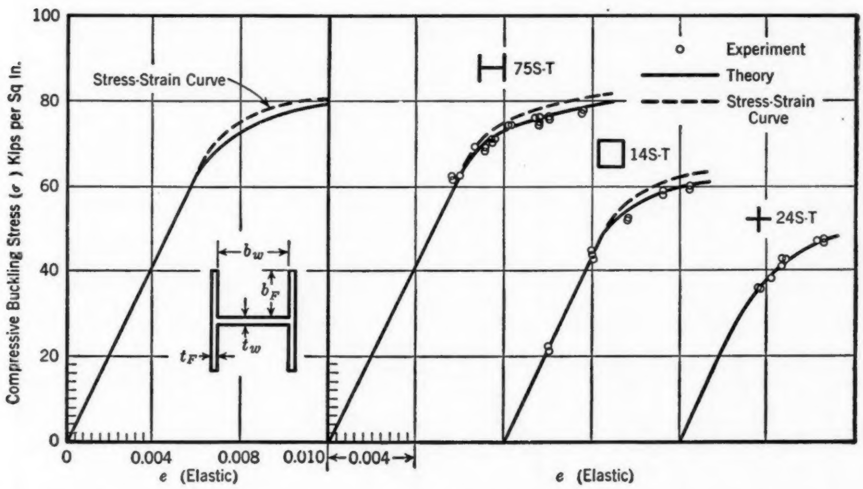


Fig. 13.—Theoretical Compressive Plate-Buckling Curve

Fig. 14.—Experimental Verification of Compressive Plate-Buckling Theory for Three Aluminum Alloy Sections

for this type of plate is the secant modulus. For the simply supported plate, as represented by the square tube, and for the H-section, the test results plot slightly below the stress-strain curve as predicted by theory, which means that the effective modulus for these specimens is slightly less than the secant modulus.

In order to show the generality of the close correlation between the test results and the stress-strain curve, the test data and stress-strain curves for a variety of aluminum and magnesium alloy H-sections, Z-sections, and channel-sections are shown in Fig. 15.[37] Because the average stress at maximum load,

[36] "Plastic Buckling of Simply Supported Compression Plates," by Richard A. Pride and George J. Heimerl, *Technical Note No. 1817*, NACA, 1949.

[37] "Determination of Plate Compressive Strengths," by George J. Heimerl, *Technical Note No. 1480*, NACA, 1947.

$\bar{\sigma}_{max}$, was also obtained, it has been included although only the buckling stress σ_c is of interest at this point. Two stress-strain curves are shown for each material to indicate the limits of variation in material properties found to exist for the sections tested. The test results for the **H**-sections fall somewhat

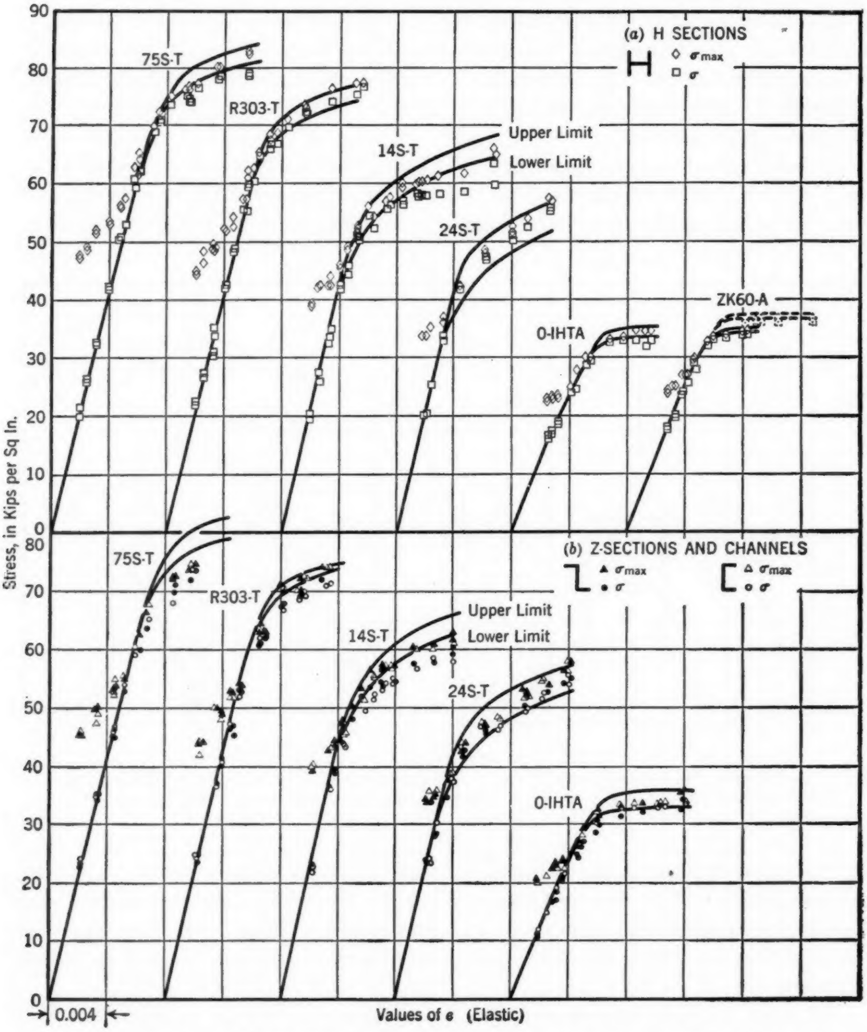


FIG. 15.—TEST DATA FOR COMPRESSIVE BUCKLING STRESS σ_c, AND THE AVERAGE STRESS AT MAXIMUM LOAD $\bar{\sigma}_{max}$ FOR EXTRUDED ALUMINUM AND MAGNESIUM ALLOYS

closer to the stress-strain curve than do those for the **Z**-sections and channels. This is probably to be expected since the **H**-sections contain a higher percentage of flange material, and the stress-strain curve is theoretically correct only for a hinged flange.

In calculating the abscissa e (elastic) for each test specimen, an uncertainty of the order of magnitude of the plate thickness arose in measuring the plate widths. Therefore, a system of dimensioning was chosen for each type of specimen such that calculated and experimental values of the buckling stress were in good agreement in the elastic range. The same system was then also used in the plastic range. For the H-sections, Z-sections, channels, and square-tube sections, the use of inside face dimensions proved to be best. For the cruciforms, however, centerline dimensions were found to be preferable.

Shear.—Fig. 16, experimental data[38] are compared with the theoretical curve of Fig. 10(b). The shear stress-strain curve, added for comparison, was deduced from the compressive stress-strain curve by dividing the compressive stress by $\sqrt{3}$ to obtain the shear stress and by multiplying the compressive strain by $\sqrt{3}$ to obtain the shear strain.

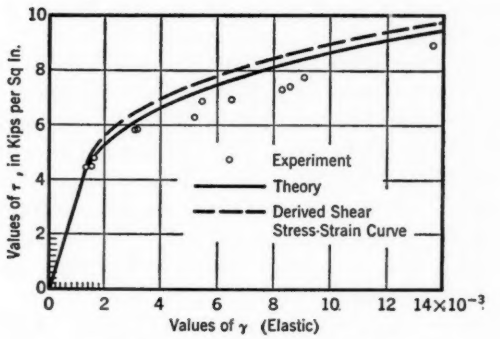


FIG. 16.—EXPERIMENTAL VERIFICATION OF SHEAR PLATE-BUCKLING THEORY

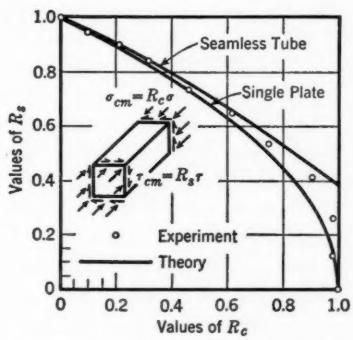


FIG. 17.—EXPERIMENTAL VERIFICATION OF THE THEORY FOR THE BUCKLING OF A PLATE UNDER COMBINED SHEAR AND LONGITUDINAL COMPRESSION

Combined Compression and Shear.—Critical combinations of torque and compression obtained in elastic buckling tests of a long square tube[39] are plotted in stress-ratio form in Fig. 17. These tubes were fabricated from four single plates connected by four corner angles. In Fig. 17 test results are compared with the parabolic interaction curve from Fig. 7(a) that should apply if the corner angles were heavy enough to isolate the walls so that they would buckle independently as four clamped plates. Since infinitely stiff angles would be required to accomplish this, the theoretical curve for a seamless square tube which takes into account the effect of adjoining walls is also included.[40] The test points, which represent averaged data from the four walls and two directions of torque, lie between the two theoretical curves.

[38] "Critical Shear Stress of Plates Above the Proportional Limit," by George Gerard, *Journal of Applied Mechanics*, March, 1948, *Transactions*, ASME, Vol. 70, p. 7.

[39] "Buckling Test of Flat Rectangular Plates Under Combined Shear and Longitudinal Compression," by Roger W. Peters, *Technical Note No. 1750*, NACA, 1948.

[40] "Buckling of a Long Square Tube in Torsion and Compression," by Bernard Budiansky, Manuel Stein, and Arthur C Gilbert, *Technical Note No. 1751*, NACA, 1948.

Maximum Strength in Compression

An accurate knowledge of the maximum strength of a structure is extremely important from the standpoint of both safety and economy. The second factor is especially important in aircraft, since the presence of unnecessary strength, and therefore unnecessary weight, will raise the operating costs. The introduction of the all-metal airplane has stimulated many studies of the maximum strength that plates develop after buckling.

Although maximum strength does not properly come under the subject of buckling, the importance of the subject and the close relationship observed between maximum strength and buckling strength warrant its discussion. The study of maximum strength in shear, through the medium of webs with transverse stiffeners, is too large a field to discuss except to call attention to some research in the field.[41] Only maximum strength in compression will be considered. The fact that the maximum strength of compressed plates may greatly exceed the buckling stress was first brought to the attention of American engineers by extensive tests at the National Bureau of Standards.[42]

Physical Action and Significant Parameters.—A simply supported plate would never fail in compression if the material remained perfectly elastic at all stresses. As the loading proceeds beyond buckling, the fraction of the total load taken by the central longitudinal fibers becomes less and less because of their buckled condition. The load shirked by the central fibers is carried by the less buckled fibers near the edges. Because of the finite strength of actual materials, a maximum load is reached when the load refused by the central fibers can no longer be taken by the edge fibers due to the fact that they are already working at stresses in the plastic region.

A precise theoretical analysis of the maximum strength of a plate would have to take into account large deflections and inelastic behavior of the material, both of which introduce complexities of nonlinearity. Such a theoretical analysis was not attempted until 1947, and then only for a relatively simple structure—the hinged flange.[43]

Even in the absence of a precise analysis, the parameters that might be significant in expressing the maximum compressive strength of a plate can be deduced from theoretical studies of the load-carrying capacity of purely elastic plates after buckling.

The first approximate analysis of this nature was given by Theodor von Kármán, M. ASCE[44] and summarized by S. P. Timoshenko.[45] Mr. von Kármán assumed that two longitudinal edge strips of combined width $m\,b$ carry the entire compression load, whereas the central strip of width $(1-\ m)b$ is free from stress. The maximum load is assumed to be reached when the

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[41] "Strength Analysis of Stiffened Beam Webs," by Paul Kuhn and James P. Peterson, *Technical Note No. 1364*, NACA, 1947.

[42] "Strength of Rectangular Flat Plates under Edge Compression," by L. Schuman and G. Back, *Report No. 356*, NACA, 1930.

[43] "The Compressive Strength of Flanges," by Elbridge Z. Stowell, *Technical Note No. 2020*, NACA, 1950.

[44] Strength of Thin Plates in Compression," by T. von Kármán, E. E. Sechler, and L. H. Donnell, *Transactions*, American Society of Mechanical Engineers, Vol. 54, 1932, p. 53.

[45] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 395.

uniform stress in the edge strips becomes equal to the yield stress of the material, σ_{cy}. The relationship that results from this analysis can be written in the form

$$\frac{\sigma}{\sigma_{\max}} = \sqrt{\frac{\sigma}{\sigma_{cy}}} \quad \ldots\ldots\ldots\ldots (16)$$

In later, more elaborate investigations[46,47] the same assumption (that the two edge strips act at a uniform stress σ_{edge}) was made, but it was further assumed that the stress in the central strip remains equal to its buckling value σ_c. The results of such studies can be put in the form

$$\bar{\sigma} = m\,\sigma_{\text{edge}} + (1 - m)\,\sigma \quad \ldots\ldots\ldots\ldots (17)$$

or

$$\frac{\sigma}{\sigma} = \frac{\dfrac{\sigma}{\sigma_{\text{edge}}}}{m + (1 - m)\,\dfrac{\sigma}{\sigma_{\text{edge}}}} \quad \ldots\ldots\ldots\ldots (18)$$

in which: $\bar{\sigma}$ is the average compressive stress; σ_{edge} is the compressive stress at the edge; σ_c is the critical buckling stress; and m is a constant, independent of the load, with a value between zero and unity.

If (as in Mr. von Kármán's analysis) the maximum strength is assumed to occur when σ_{edge} reaches some high value (say, the compressive yield stress σ_{cy}), then Eq. 18 can be written as

$$\frac{\sigma}{\sigma_{\max}} = \frac{\dfrac{\sigma}{\sigma_{cy}}}{m + (1 - m)\,\dfrac{\sigma}{\sigma_{cy}}} \quad \ldots\ldots\ldots\ldots (19)$$

Eqs. 18 and 19, derived from different assumptions, are both based on elastic theory, and indicate that $\frac{\sigma}{\sigma_{\max}}$ and $\frac{\sigma}{\sigma_{cy}}$ might be significant parameters. Therefore, the results of the maximum strength tests were plotted in terms of these parameters.

Correlation Between Buckling Strength and Maximum Strength.—Maximum-strength data are plotted in Fig. 18 for various sections of aluminum and magnesium alloys. The data[37] are plotted in the form of curves of $\frac{\sigma}{\sigma_{\max}}$ versus $\frac{\sigma}{\sigma_{cy}}$.

For the cruciform section, the test data show good agreement with a theoretical curve calculated for a hinged flange utilizing the same plasticity relationships as those used in the plastic buckling analysis (see Appendix I). In the tests, the juncture of the four flanges of the cruciform section was observed to remain straight to and beyond the maximum load. The cruciform, therefore, may truly be regarded as four hinged flanges at maximum load as well as at the buckling load.

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[46] "The Apparent Width of the Plate in Compression," by Karl Marguerre, *Technical Memorandum No. 833*, NACA, 1942.

[47] "Bending of Rectangular Plates with Large Deflections," by Samuel Levy, *Technical Note No. 846*, NACA, 1942.

In Fig. 18, the calculated dashed curve for maximum strength of flanges just described is also superimposed on the experimental data for the H-sections Z-sections, channels, and square-tube sections. Although this curve strictly applies only to a hinged flange, it seems to describe fairly well the results for

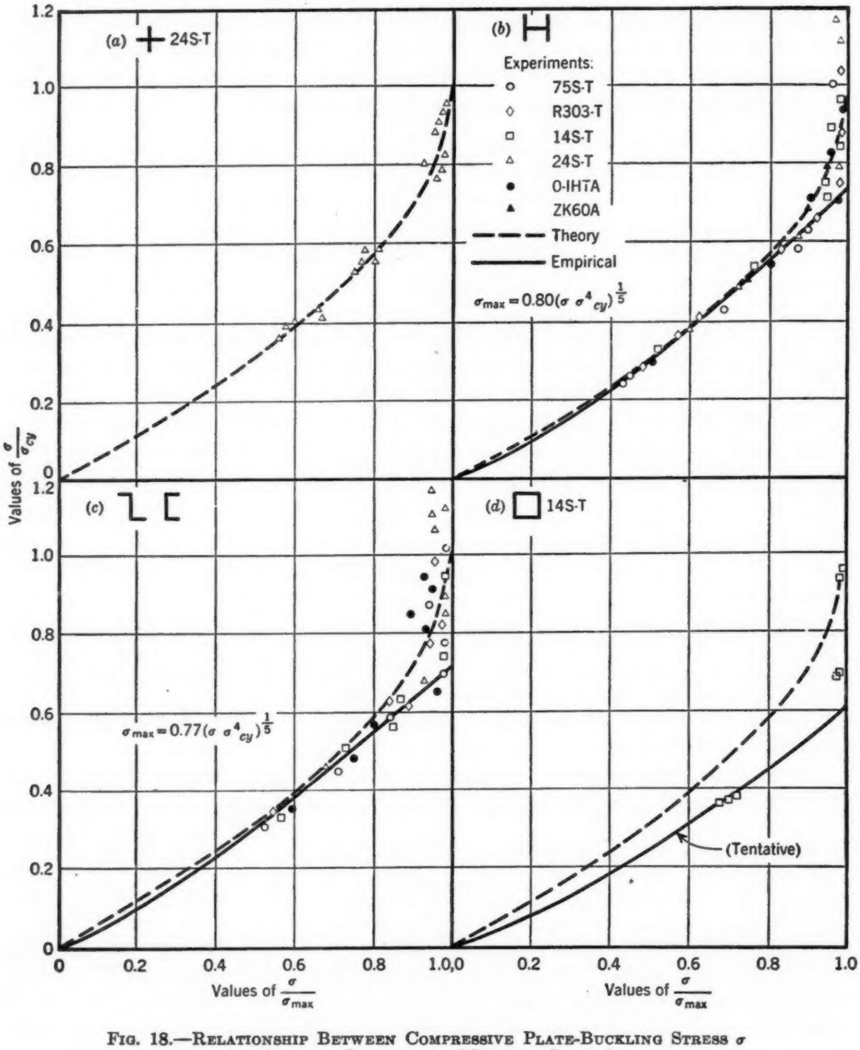

FIG. 18.—RELATIONSHIP BETWEEN COMPRESSIVE PLATE-BUCKLING STRESS σ AND AVERAGE STRESS AT A MAXIMUM LOAD $\bar{\sigma}_{max}$

these sections, for which flanges were primarily responsible for instability; but it misses widely the few results for the square-tube section, which does not possess flanges. Empirical curves that apply specifically to H-sections, Z-sections, and channel sections are also shown. The equations for these

curves provide simple analytical expressions for engineering use. Although the data are very few for the square tube, and indicate a lower average stress at maximum load for the same ratio of buckling stress to yield stress $\frac{\sigma}{\sigma_{cy}}$, there are enough data to indicate that a similar empirical curve probably applies. One such curve, tentatively proposed, is shown in Fig. 18.

The grouping of all the experimental points close to a value $\frac{\sigma}{\sigma_{\max}} = 1$, when the ratio $\frac{\sigma}{\sigma_{cy}}$ is greater than about 3/4, implies that for practical engineering purposes buckling at a stress greater than 3/4 of the yield stress is equivalent to failure. If the buckling stress is lower than about 3/4 the yield stress, the relation between the buckling stress and the maximum strength is nearly the same for all the flanged sections and all the materials. A similar but different relationship would be expected to hold for the square tubes. The statements in this paragraph apply to metals of the type of aluminum or magnesium alloys but may have to be modified for members made of mild steel, in view of the basically different shape of the stress-strain curve of this material.

It should be noted that the data in Fig. 18 are for integral plate sections with relatively sharp corners and without corner angles. The presence of curvature of connections angles at the corners can be expected to alter, somewhat, the relationships obtained.

Concluding Remarks

Charts have been presented for the calculation of elastic and plastic buckling stresses of flat unstiffened plates in compression, shear, and combined compression and shear, and for integral flat-plate sections in compression. Correlations have been shown between buckling strength and maximum strength for the integral flat-plate sections. Theoretical methods and experimental techniques for conducting buckling investigations are indicated briefly in Appendixes I and II.

The agreement shown between theory and experiment indicates that the status of the problem of calculating flat-plate buckling strength is satisfactory. Buckling stresses can be calculated with engineering accuracy in both the elastic and plastic stress ranges.

A survey of the various solutions available indicates that the different types of loading have not all been studied to the same extent. For most cases the solutions have stopped short of the plastic range and many have been concerned with only the simpler edge conditions. However, the foundations have now been laid for the gaps to be filled at least by extrapolation. In addition, solutions for a number of loading conditions other than those included in the paper are available in the literature or are in the process of being obtained.

A field that needs much work to lay a comparable foundation is that concerned with plate assemblies which are not integral, but fabricated with the aid of angles and other connections. Two uncertainties confront the analyst in his study of such plate assemblies. The first is the uncertainty as to the widths of the component plates due to the relatively large size of the con-

nection angles; the second is the uncertainty as to the degree of continuity provided by the connections. The first uncertainty is also present to some extent in formed plate sections when the curved material created by forming is an appreciable part of the total cross-sectional area.

Additional problems include the determination of the size of permanent buckles resulting from loads above the buckling load, the effect of a permanent buckle on the ultimate strength, the fatigue life of a plate under repeated buckling, and the interaction of plate buckling with other types of instability involving the structure as a whole.

Acknowledgment

Valuable advice and assistance have been received in the preparation of this paper from associates of the writers in the Structures Research Division of NACA

Appendix I.—Elastic Theoretical Buckling Analysis

Differential Equation Method.—An exact mathematical analysis to determine the buckling strength of a flat plate involves the integration of the partial differential equation[13]

$$\frac{E\,t^3}{12\,(1-\mu^2)}\left[\frac{\partial^4 w}{\partial x^4}+2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right] = -\,t\left[\sigma_x\,\frac{\partial^2 w}{\partial x^2}+2\,\tau\,\frac{\partial^2 w}{\partial x\,\partial y}+\sigma_y\,\frac{\partial^2 w}{\partial y^2}\right] \quad (20)$$

in which w is the lateral deflection of the plate; x and y are the length and width coordinates respectively; E is Young's modulus; μ is Poisson's ratio; t is the thickness; σ_x and σ_y are the middle plane compressive stresses in the x-direction and y-direction, respectively; and τ is the middle-plane shear in these directions. The stresses σ_x, σ_y, and τ are known from the external loading at buckling.

Detailed methods of solving the differential equations are too numerous to include in the paper.[48,49]

Energy Method.—Also useful to the calculation of buckling stresses is the energy method. The "potential energy" can be written as follows for plates with edges either free, simply supported, or clamped:[14]

$$\Delta W = \frac{E\,t^3}{24\,(1-\mu^2)}\int\!\!\int\left\{\sigma_x\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy - \frac{t}{2}\int\!\!\int\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2+2\,\tau\,\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\sigma_y\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy \quad (21)$$

[48] "Mathematical Theory of Elasticity," by I. S. Sokolnikoff, McGraw-Hill Book Co., Inc., New York, N. Y., 1946, Chapter V.

[49] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, Chapter VII.

in which the integration is performed over the length and width of the plate. The first integral in Eq. 21 represent the strain energy gain of the plate due to the bending and twist that occur during buckling. The second integral represents an energy loss associated with the middle-plane stresses due to the lateral deflection. If the plate edges are held fixed during buckling this integral represents a reduction in middle-plane stretching energy; if the edges may shift with respect to one another, it represents the work of the external loading.

The potential energy expression, Eq. 21 is useful by virtue of the fact that, of all functions w satisfying the geometric boundary conditions (conditions on deflection and slope), ΔW will be a minimum (equal to zero) for that function which also satisfies the differential equation.[48] Thus, of several functions w satisfying the geometric boundary conditions but not necessarily the differential equation, the one which satisfies the differential equation best is the one for which ΔW is smallest. A special development of the energy method is due to Lord Rayleigh[50] and Walter Ritz.[51]

Although buckling stresses obtained by this approximate method may be very accurate, they are generally higher than the exact stresses. Thus, the energy method furnishes an upper limit to the buckling stress unless, by chance, the deflection pattern chosen for w is the true one—that is, the one that can satisfy the differential Eq. 20 exactly.

Lagrangian Multiplier Method.—By a slight but fundamental revision in the energy method, conservative results can be obtained. The modification consists in using a complete set of functions for w that does not satisfy the geometric boundary conditions term by term but is capable of satisfying them as a whole. If the set of functions is so chosen that they satisfy the boundary conditions only approximately, a lower limit to the buckling stress may be obtained.

The technique of relaxing boundary conditions was first used for buckling problems by E. Trefftz.[52] The method is described and refined by Bernard Budiansky, Pai C. Hu, and Robert W. Connor[53,54] and it is called the Lagrangian multiplier method. It is presented in such a way that both the upper-limit and lower-limit solutions can be obtained from the same mathematical setup.

Plastic Theoretical Buckling Analysis

Buckling stresses obtained by the methods described in the first part of this appendix are correct only if they fall in the elastic range—that is, below the proportional limit of the plate material. This follows from the assumption of perfect elasticity upon which the derivation of Eqs. 20 and 21 is based.

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[50] "The Theory of Sound," by John William Strutt, Baron Rayleigh, Macmillan Co., New York, N. Y., 1929, Chapter IV.

[51] "Über eine Neue Methode Zur Lösung gewisser Variationprobleme der Mathematischen Physik," by Walter Ritz, *Journal für reine und angewandte Mathematik*, Vol. 135, 1909, p. 1.

[52] "Ein Gegenstück zum Ritzschen Verfahren," by E. Trefftz, *Proceedings*, 2d International Congress for Applied Mechanics, Zürich, 1926, pp. 131–137.

[53] "The Lagrangian Multiplier Method of Finding Upper and Lower Limits to Critical Stresses of Clamped Plates," by Bernard Budiansky and Pai C. Hu, *Technical Note No. 1103*, NACA, 1946.

[54] "Notes on the Lagrangian Multiplier Method in Elastic-Stability Analysis," by Bernard Budiansky, Pai C. Hu, and Robert W. Connor, *Technical Note No. 1558*, NACA, 1948.

Equations to replace Eqs. 20 and 21 for buckling in the plastic range have been proposed by various investigators.

The problem of plastic buckling analysis of plates is much more difficult than the corresponding problem for columns. For a long time, therefore, equations to replace Eq. 20 in the plastic range were written as intuitive generalizations of the corresponding plastic column buckling equation.[49,55] Eventually, more refined attempts were made to solve the plastic buckling problem through the use of theories for the plastic behavior of materials under combined stresses. The first attempts were made by P. P. Bijlaard,[31] M. ASCE, and A. A. Ilyushin.[56] The latter's analysis was modified by one of the writers[32,33,34] to incorporate the Shanley continuous-loading principle.[57,58] The resulting stresses show good agreement with experiment. This type of theory has received theoretical justification in the literature.[59]

The method of analysis proceeds from the assumption that in the plastic domain a unique stress-strain curve exists for combined loading that is a simple generalization of the stress-strain curve for a single loading. The existence of such a curve has been established approximately by experiment (and has also been justified theoretically) for the case in which the ratios of the various stresses are maintained constant as the loading proceeds[60]; unloading occurs elastically as is the usual case.

By making use of the generalized stress-strain curve and the corresponding stress-strain relations, equations for the plastic range were obtained to replace Eqs. 20 and 21, which hold in the elastic range.[32] Solutions of the equations were obtained by methods analogous to those in the corresponding elastic cases.

APPENDIX II.—EXPERIMENTAL BUCKLING ANALYSIS

Test specimens.—Most of the experimental work conducted to check the theoretical results thus far described has been concerned with loading in com-

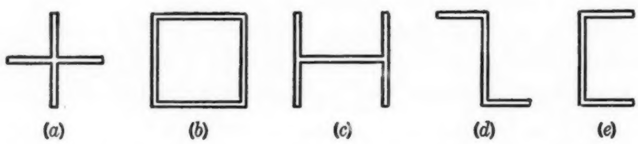

FIG. 19.—CROSS SECTIONS OF SPECIMENS USED IN COMPRESSIVE PLATE-BUCKLING TESTS

pression. The test specimens used to obtain the data in Figs. 14 and 15 consisted of so-called cruciforms (+-sections), square tubes, and **H**-sections, **Z**-sections, and channel sections as shown in Fig. 19. All these sections

[55] "Theorie und Berechnung der Eisernen Brücken," by Friedrich Bleich, Springer, Berlin, 1924, p. 216.

[56] "The Elasto-Plastic Stability of Plates," by A. A. Ilyushin, *Technical Memorandum No. 1188*, NACA, 1947.

[57] "The Column Paradox," by F. R. Shanley, *Journal of the Aeronautical Sciences*, Vol. 13, No. 12, December, 1946, p. 678.

[58] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, Vol. 14, No. 5, May, 1947, p. 261.

[59] "Theories of Plastic Buckling," by S. B. Batdorf, *Journal of the Aeronautical Sciences*, Vol. 16, No. 7, July, 1949, p. 405.

[60] "Plastichnost," by A. A. Ilyushin, OGIZ, Moskva-Leningrad, 1948, pp. 69, 75, and 117.

were made by the extrusion process with the exception of the square tubes which were drawn.

The tests on the cruciforms (Fig. 19) and square tubes were for the purpose of obtaining data for single plates rather than for plate sections. In effect, the cruciform is composed of four flanges mutually supported along a common edge, as can be seen in Fig. 20(*a*). The square tube of constant wall thickness shown in Fig. 20(*b*) is similarly an integral assembly of four plates mutually supporting each other along both side edges.

The lengths were chosen short enough so that plate buckling would occur with the corners remaining straight, but not so short as to incur any appreciable increase in strength associated with short lengths. For the cruciform section the wave lengths developed on buckling were much longer in terms of plate widths than for the other sections as indicated in Fig. 20.

The test data shown in Fig. 16 for shear were obtained by George Gerard[38] from tests on panels of thin sheet material in a loading frame. Tests to check Fig. 7(*a*)—the interaction curve for combined compression and shear—have been made only in the elastic range.[39] The specimen used was a long square tube consisting of four thin flat plates joined by relatively heavy angles at the corners.

Compressive Stress-Strain Tests.—Compressive stress-strain curves, which are necessary in order to correlate the material properties with the plate compressive strength and for use in applying plasticity theory were obtained chiefly from tests of single-thickness specimens cut from the elements of the sections from which the buckling specimens were taken. The single-thickness specimens were tested in a modified form of the Montgomery-Templin type of compression fixture, utilizing longitudinally grooved supporting plates in place of rollers used in the original fixture. This type of fixture and associated techniques are described in several publications.[61]

Detection of Buckling.—In the theory of buckling, a plate is assumed to be perfectly flat; thus lateral deflection under edge load will not occur until a sharply defined critical load is reached. In practical tests, however, plates (no matter how carefully they are fabricated) have some slight initial deviations from flatness which grow with increase in load. Thus ideal buckling is not realized, and the problem arises of how best to interpret experimental data so as to obtain the buckling loads.

One procedure that has been proposed for determining experimental plate-buckling loads is based on an extension of the Southwell plot method used for columns.[62,63,64] To be successful, the method requires that the theory of small deflection be applicable and, at the same time, that the loads at which measurements are taken be close to the theoretical buckling load. Although these two conditions are attainable for columns, they are very often mutually

[61] "Determination and Presentation of Compressive Stress-Strain Data for Thin Sheet Metal," by Walter Ramberg and James A. Miller, *Journal of the Aeronautical Sciences*, Vol. 13, No. 11, November, 1946, p. 569.

[62] "On the Analysis of Experimental Observations in Problems of Elastic Stability," by R. V. Southwell, *Proceedings*, Royal Society of London, Section A, Vol. 135, April, 1932.

[63] "Generalized Analysis of Experimental Observations in Problems of Elastic Stability," by Eugene E. Lundquist, *Technical Note No. 658*, NACA, 1938.

[64] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, pp. 321 and 401.

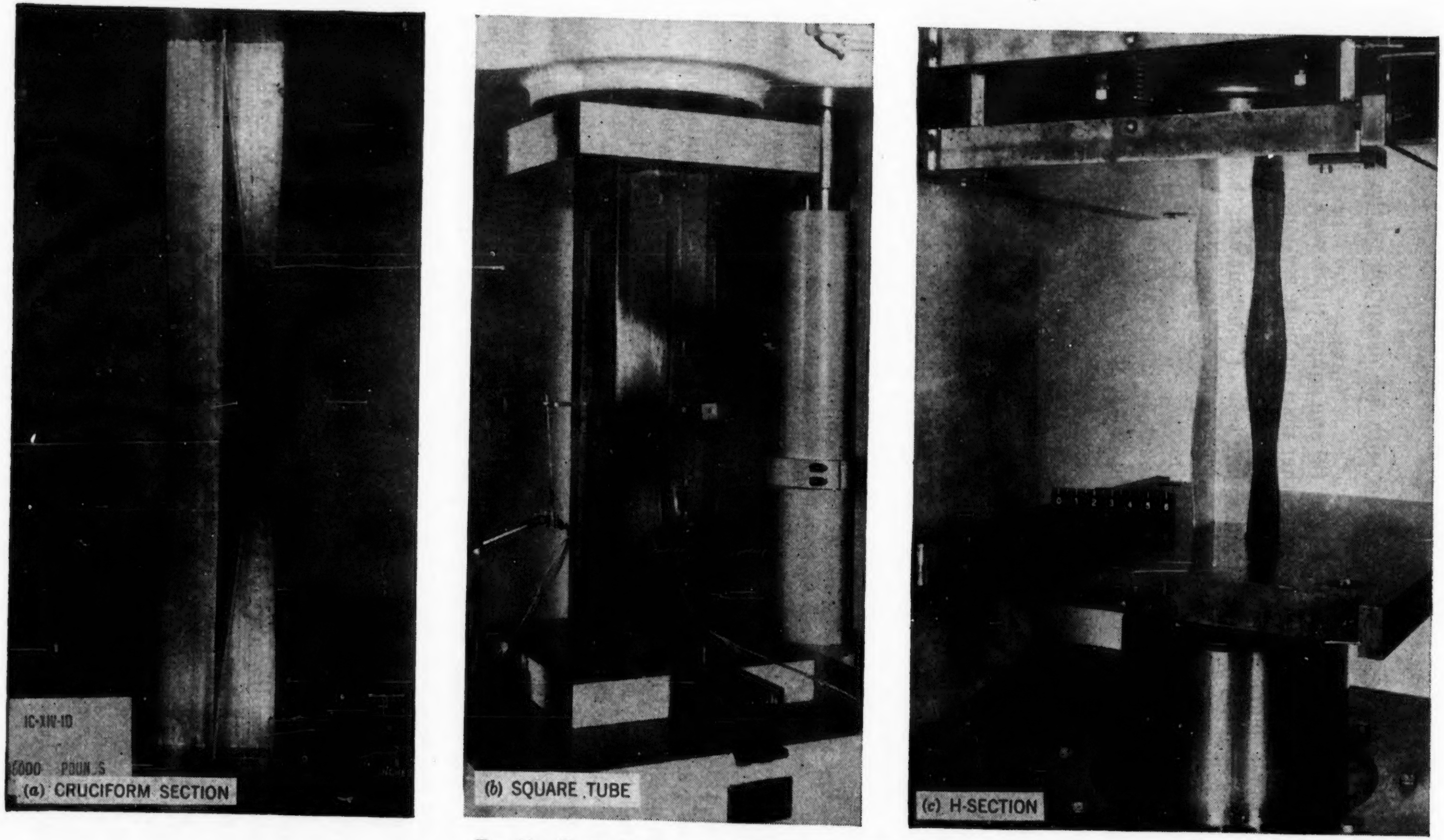

Fig. 20.—Plate Buckling of Sections in Compression

contradictory for plates. In addition, precise measurements of strain or deflection were required. These difficulties, coupled with the fact that the method is strictly applicable to plates only in the elastic range, suggested that a more direct and simple engineering way of obtaining an experimental buckling load should be adopted.

Consequently, two more practical methods of defining experimental buckling have been considered. According to one—the strain-reversal method—the buckling load was taken as the load at which the strain on the convex side of a buckle stopped increasing and started decreasing. According to the other (the so-called "top-of-the-knee" method) the buckling load was the load corresponding to the top of the knee of a curve of load versus lateral deflection.

In order to assess the amount by which experimental buckling stresses determined by the strain-reversal or top-of-the knee methods might differ from the theoretical buckling stress of a perfectly flat plate, a theoretical study was made, by means of elastic large-deflection theory, of the behavior under edge compression of a simply supported square plate with slight initial deviations from flatness.[65] This study indicated that the top-of-the-knee method gives buckling loads below, but closer to, the theoretical load than the strain-reversal method.

Certain practical difficulties in using the strain-reversal method further strengthened the case for the top-of-the-knee method. These included inability to predict accurately the location of a buckle crest and, in some cases, the absence of strain reversal, particularly in the plastic buckling range. For these reasons, the experimental buckling stresses for compression given in this paper were determined by the top-of-the-knee method. For most of the tests, the load and deflection were recorded graphically.

The shear buckling stresses of Fig. 16 were based on the so-called strain-difference method.[38] The load readings were plotted against buckle curvature. The critical stress was defined by the intersection of two straight lines through the plotted test points. The buckling stresses obtained by this method generally lie between those given by the top-of-the-knee and strain-reversal methods.

For tests of the tube in combined compression and torsion to check the theoretical interaction curve of Fig. 7(a), the strain-reversal method was used to detect buckling. Although this would cause the actual buckling stresses to be low, conversion of the results to stress-ratio form was judged to eliminate most of the error due to using this definition of buckling.

[65] "Effect of Small Deviations from Flatness on Effective Width and Buckling of Plates in Compression," by Pai C. Hu, Eugene E. Lundquist, and S. B. Batdorf, *Technical Note No. 1124*, NACA, 1946.